Implications of the Daya Bay observation of θ_{13} on the leptonic flavor mixing structure and CP violation

Zhi-zhong Xing *

Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China

Abstract

The Daya Bay Collaboration has recently reported its first $\overline{\nu}_e \to \overline{\nu}_e$ oscillation result which points to $\theta_{13} \simeq 8.8^\circ \pm 0.8^\circ$ (best-fit $\pm 1\sigma$ range) or $\theta_{13} \neq 0^\circ$ at the 5.2σ level. The fact that this smallest neutrino mixing angle is not strongly suppressed motivates us to look into the underlying structure of lepton flavor mixing and CP violation. Two phenomenological strategies are outlined: (1) the lepton flavor mixing matrix U consists of a constant leading term U_0 and a small perturbation term ΔU; and (2) the mixing angles of U are associated with the lepton mass ratios. Some typical patterns of U_0 are reexamined by constraining their respective perturbations with current experimental data. We illustrate a few possible ways to minimally correct U_0 in order to fit the observed values of three mixing angles. We point out that the structure of U may exhibit an approximate μ-τ permutation symmetry in modulus, and reiterate the geometrical description of CP violation in terms of the leptonic unitarity triangles. The salient features of nine distinct parametrizations of U are summarized, and its Wolfenstein-like expansion is presented by taking U_0 to be the democratic mixing pattern.

PACS number(s): 14.60.Pq, 13.10.+q, 25.30.Pt

Typeset using REVTeX

*E-mail: xingzz@ihep.ac.cn

I. INTRODUCTION

Thanks to a number of well-done solar, atmospheric, reactor and accelerator neutrino oscillation experiments, we are now convinced that three known neutrinos have finite masses and one lepton flavor can convert to another [1]. The phenomenon of lepton flavor mixing at low energies is effectively described by a 3×3 matrix U, the so-called Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix [2], in the weak charged-current interactions:

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{\left(e \quad \mu \quad \tau \right)_{\rm L}} \gamma^\mu \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} W^-_\mu + {\rm h.c.} \; . \tag{1}$$

Given the unitarity of U, it can be parametrized in terms of three angles and three phases:

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{13} s_{23} e^{i\delta} & c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta} & c_{13} s_{23} \\ s_{12} s_{23} - c_{12} s_{13} c_{23} e^{i\delta} & -c_{12} s_{23} - s_{12} s_{13} c_{23} e^{i\delta} & c_{13} c_{23} \end{pmatrix} P_\nu \; , \tag{2}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ (for $ij = 12, 13, 23$), and $P_\nu = {\rm Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$ which is physically relevant if massive neutrinos are the Majorana particles. A global analysis of the available neutrino oscillation data [3] points to $\theta_{12} \simeq 34^\circ$ and $\theta_{23} \simeq 45^\circ$, much larger than the Cabibbo angle $\vartheta_{\rm C} \simeq 13^\circ$ in the Cabibbo-Kobayashi-Maskawa (CKM) quark flavor mixing matrix V [4]. The third mixing angle θ_{13} is expected to be smaller than $\vartheta_{\rm C}$, and its central value might be around 8° [5] as hinted by the preliminary T2K [6], MINOS [7] and Double Chooz [8] data. Three CP-violating phases of U remain unknown at this stage, but one of them (i.e., the Dirac phase δ) will be measured in the forthcoming long-baseline neutrino oscillation experiments.

The Daya Bay Collaboration has recently made a breakthrough in the measurement of θ_{13} from the reactor $\overline{\nu}_e \to \overline{\nu}_e$ oscillations [9]. The best-fit ($\pm 1\sigma$ range) result is

$$\sin^2 2\theta_{13} = 0.092 \pm 0.016 ({\rm stat}) \pm 0.005 ({\rm syst}) \; , \tag{3}$$

which is equivalent to $\theta_{13} \simeq 8.8^\circ \pm 0.8^\circ$ or $\theta_{13} \neq 0^\circ$ at the 5.2σ level. This very encouraging observation convinces us that the smallest neutrino mixing angle is not really small and the MNSP matrix U is not strongly hierarchical. We are therefore motivated to study the underlying structure of lepton flavor mixing and CP violation. In fact, U has been conjectured to have the following structure for a quite long time [10]:

$$U = \left(U_0 + \Delta U \right) P_\nu \; , \tag{4}$$

in which the leading term U_0 is a constant matrix responsible for two larger mixing angles θ_{12} and θ_{23}, and the next-to-leading term ΔU is a perturbation matrix responsible for both the smallest mixing angle θ_{13} and the Dirac CP-violating phase δ. So far a lot of flavor symmetries have been brought into exercise to derive U_0, while ΔU might originate from either an explicit flavor symmetry breaking scenario or some finite quantum corrections at a given energy scale or from a superhigh-energy scale to the electroweak scale. In view of the new and robust Daya Bay result for θ_{13}, we are immediately concerned about two burning issues of the day in the phenomenology of neutrino physics:

- If the essential structure of lepton flavor mixing is really revealed by Eq. (4), can there be a natural pattern of U_0 accompanied by a natural perturbation matrix ΔU?
- If the main part of the MNSP matrix U is not a constant mixing matrix, what is the most straightforward way to understand the salient features of lepton flavor mixing?

In addition, we are curious about whether the structure of U has an approximate μ-τ permutation symmetry in modulus, whether leptonic CP violation is significant in neutrino oscillations, whether the other parametrizations of U besides the one in Eq. (2) are useful for describing the properties of lepton flavor mixing and CP violation, and whether there is an interesting and suggestive expansion of U as compared with the popular Wolfenstein parametrization of the CKM matrix V [11], and so on.

The purpose of this paper is to answer those easy questions and outline some possible ways to deal with those difficult ones as mentioned above. In section II we describe two phenomenological strategies towards understanding the textures of lepton mass matrices and thus the structure of lepton flavor mixing: one of them can result in Eq. (4), and the other is expected to relate the mixing angles to the lepton mass ratios. In section III we reexamine five typical patterns of U_0 (the democratic [10], bimaximal [12], tri-bimaximal [13], golden-ratio [14] and hexagonal [15] mixing patterns) by estimating their respective perturbation matrices with the help of the latest Daya Bay result for θ_{13}. Except the democratic mixing pattern, we find that the other four patterns of U_0 suffer from a common problem: the viable perturbation matrix ΔU has to be adjusted in a more or less unnatural way to make one or two of the large mixing angles of U_0 slightly modified but its smallest (vanishing) angle significantly corrected. Section IV is devoted to a brief discussion about the possible minimal perturbations to U_0. We take three interesting examples to illustrate three simple approaches for this goal. In section V we point out the conditions under which the MNSP matrix U may have an exact or approximate μ-τ permutation symmetry in modulus. The strength of leptonic CP violation is calculated, and the language of leptonic unitarity triangles is reiterated to geometrically describe CP violation. Section VI is devoted to a summary of nine topologically distinct parametrizations of U and their respective features or merits, and section VII is devoted to a Wolfenstein-like expansion of U by taking U_0 to be the democratic mixing pattern. In section VIII we first summarize the main points and results of this paper and then make some concluding remarks.

II. TWO PHENOMENOLOGICAL STRATEGIES

The MNSP matrix U actually describes a fundamental mismatch between the weak-interaction (flavor) and mass eigenstates of six leptons, or equivalently a mismatch between the diagonalizations of the charged-lepton mass matrix M_l and the effective neutrino mass matrix M_ν in a given model, no matter whether the origin of neutrino masses is attributed to the seesaw mechanisms or not [16]. Assuming massive neutrinos to be the Majorana particles, we may simply write out the leptonic mass terms as

$$-\mathcal{L}_{\rm mass} = \overline{\left(e' \quad \mu' \quad \tau' \right)_{\rm L}} \; M_l \begin{pmatrix} e' \\ \mu' \\ \tau' \end{pmatrix}_{\rm R} + \frac{1}{2} \overline{\left(\nu_e \quad \nu_\mu \quad \nu_\tau \right)_{\rm L}} \; M_\nu \begin{pmatrix} \nu_e^c \\ \nu_\mu^c \\ \nu_\tau^c \end{pmatrix}_{\rm R} + {\rm h.c.} \; , \qquad (5)$$

in which "l" stands for the flavor eigenstates of charged leptons, "c" denotes the charge-conjugated neutrino fields, and M_ν is symmetric. By means of the unitary matrices O_l, $O^\prime_l$ and O_ν, one can diagonalize M_l and M_ν through the transformations $O^\dagger_l M_l O^\prime_l = \widehat{M}_l \equiv \mathrm{Diag}\{m_e, m_\mu, m_\tau\}$ and $O^\dagger_\nu M_\nu O^*_\nu = \widehat{M}_\nu \equiv \mathrm{Diag}\{m_1, m_2, m_3\}$, respectively. Then one arrives at the lepton mass terms in terms of the mass eigenstates:

$$-\mathcal{L}^\prime_{\rm mass} = \overline{(e \quad \mu \quad \tau)_{\rm L}} \; \widehat{M}_l \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_{\rm R} + \frac{1}{2} \overline{(\nu_1 \quad \nu_2 \quad \nu_3)_{\rm L}} \; \widehat{M}_\nu \begin{pmatrix} \nu^c_1 \\ \nu^c_2 \\ \nu^c_3 \end{pmatrix}_{\rm R} + {\rm h.c.} \; . \tag{6}$$

Extending this basis transformation to the standard weak charged-current interactions, we immediately obtain Eq. (1) in which the MNSP matrix U is given by $U = O^\dagger_l O_\nu$.

The above treatment is most general at a given energy scale (e.g., the electroweak scale), but it can still provide us with the following lessons:

- The structure of lepton flavor mixing is directly determined by the structures of O_l and O_ν. Since these two unitary matrices are used to diagonalize M_l and M_ν, respectively, their structures are governed by those of M_l and M_ν, whose eigenvalues are the physical lepton masses. Therefore, we anticipate that the dimensionless flavor mixing angles of U should be certain kinds of functions whose variables include four independent mass ratios of three charged leptons and three neutrinos. Namely,

$$\theta_{ij} = f\left(\frac{m_\alpha}{m_\beta}, \frac{m_k}{m_l}, \cdots\right) \; , \tag{7}$$

  where the Greek subscripts denote the charged leptons, the Latin subscripts stand for the neutrinos, and "$\cdots$" implies other possible dimensionless parameters originating from the lepton mass matrices. Such an expectation has proved valid in the quark sector to explain why the relation $\sin\vartheta_{\rm C} \simeq \sqrt{m_d/m_s}$ works quite well and how the hierarchical structure of the CKM matrix V is related to the strong hierarchies of quark masses (i.e., $m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$) [16]. As for the phenomenon of lepton flavor mixing, it is apparently difficult to link two large mixing angles θ_{12} and θ_{23} to two small mass ratios $m_e/m_\mu \simeq 4.7 \times 10^{-3}$ and $m_\mu/m_\tau \simeq 5.9 \times 10^{-2}$ [17]. Hence one may consider to ascribe the largeness of θ_{12} and θ_{23} to a very weak hierarchy of three neutrino masses, such as the conjecture $\tan\theta_{12} \simeq \sqrt{m_1/m_2}$ [18].

- To establish a direct relation between θ_{ij} and lepton mass ratios, one has to specify the textures of M_l and M_ν by allowing some of their elements to vanish or to be vanishingly small. The most instructive example of this kind is the Fritzsch ansatz [19],

$$M_{l,\nu} = \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix} \; , \tag{8}$$

  which is able to account for current neutrino oscillation data to an acceptable degree of accuracy (e.g., $\sin\theta_{23} \simeq \sqrt{m_\mu/m_\tau} + \sqrt{m_2/m_3} \simeq 0.65$) [20]. Another well-known and phenomenologically viable example is the two-zero textures of M_ν in the basis where M_l is diagonal [21]. Note that the texture zeros of a fermion mass matrix dynamically mean that the corresponding matrix elements are sufficiently suppressed as compared with their neighboring counterparts, and they can be derived from a certain flavor symmetry in a given theoretical framework (e.g., with the help of the Froggatt-Nielson mechanism [22] or discrete flavor symmetries [23]).

- We realize that the expectation in Eq. (7) is actually in conflict with the conjecture made in Eq. (4). In other words, the leading term of the MNSP matrix U might be a constant matrix whose mixing angles are independent of the lepton mass ratios. The reason for this "conflict" is rather simple: the assumed structures of lepton flavor mixing in Eqs. (4) and (7) correspond to two different structures of lepton mass matrices. As we have pointed out above, the direct dependence of θ_{ij} on m_α/m_β and m_k/m_l is usually a direct consequence of the texture zeros of M_l and (or) M_ν. In contrast, a constant flavor mixing pattern U_0 may arise from some special textures of M_l and (or) M_ν whose entries have certain kinds of linear correlations or equalities. For instance, the texture [24]

$$M_\nu = \begin{pmatrix} b+c & -b & -c \\ -b & a+b & -a \\ -c & -a & a+c \end{pmatrix} \tag{9}$$

  assures O_ν to be of the tri-bimaximal mixing pattern. This neutrino mass matrix has no zero entries, but its nine elements satisfy the sum rules $(M_\nu)_{i1}+(M_\nu)_{i2}+(M_\nu)_{i3}=0$ and $(M_\nu)_{1j}+(M_\nu)_{2j}+(M_\nu)_{3j}=0$ (for $i,j=1,2,3$). Such correlative relations are similar to those texture zeros in the sense that both of them may reduce the number of free parameters associated with lepton mass matrices, making some predictions for the lepton flavor mixing angles technically possible.

- It is well known that the special textures of M_l and M_ν like that in Eq. (9) can easily be derived from certain discrete flavor symmetries (e.g., A_4 or S_4) [25]. That is why Eq. (4) formally summarizes a large class of lepton flavor mixing patterns in which the leading terms are constant matrices originating from some underlying flavor symmetries. The fact that θ_{13} is not very small poses a meaningful question to us today: can this mixing angle naturally be generated from the perturbation matrix ΔU? The answer to this question is certainly dependent upon the form of U_0 in the flavor symmetry limit. We shall reexamine five typical patterns of U_0 in the subsequent section to get a feeling of the respective structures of ΔU which can be constrained by current experimental data on neutrino oscillations.

In short, one may try to understand the structure of the MNSP matrix U by following two phenomenological strategies: one is to explore possible relations between the flavor mixing angles and the lepton mass ratios, and the other is to investigate possible constant patterns of lepton flavor mixing as the leading-order effects. We have seen that the former possibility essentially points to some vanishing (or vanishingly small) entries of M_l and M_ν, while the latter possibility apparently indicates some equalities or linear correlations among the entries of M_l or M_ν. In both cases the underlying flavor symmetries play a crucial role in deriving the structures of lepton mass matrices which finally determine the structure of lepton flavor mixing. Of course, how to pin down the correct flavor symmetries remains an open question.

III. FIVE PATTERNS OF THE MNSP MATRIX

For the sake of simplicity, we typically take $\theta_{12} \simeq 34^\circ$, $\theta_{13} \simeq 9^\circ$ and $\theta_{23} \simeq 45^\circ$ as our inputs to fix the primary structure of the MNSP matrix U. Then we arrive at

$$U = \begin{pmatrix} 0.819 & 0.552 & 0.156 e^{-i\delta} \\ -0.395 - 0.092 e^{i\delta} & 0.586 - 0.062 e^{i\delta} & 0.698 \\ 0.395 - 0.092 e^{i\delta} & -0.586 - 0.062 e^{i\delta} & 0.698 \end{pmatrix} P_\nu \; . \tag{10}$$

It makes sense to compare a constant mixing pattern U_0 with the observed pattern of U in Eq. (10), such that one may estimate the structure of the corresponding perturbation matrix ΔU. Let us consider five well-known patterns of U_0 for illustration.

(1) The democratic mixing pattern of lepton flavors [10]:

$$U_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} , \tag{11}$$

whose three mixing angles are $\theta_{12}^{(0)} = 45^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = \arctan(\sqrt{2}) \simeq 54.7^\circ$ in the standard parametrization as given in Eq. (2). With the help of Eq. (10), we immediately obtain the form of $\Delta U = U P_\nu^\dagger - U_0$ as follows:

$$\Delta U = \begin{pmatrix} 0.112 & -0.155 & 0.156 e^{-i\delta} \\ 0.013 - 0.092 e^{i\delta} & 0.178 - 0.062 e^{i\delta} & -0.118 \\ -0.182 - 0.092 e^{i\delta} & -0.009 - 0.062 e^{i\delta} & 0.121 \end{pmatrix} . \tag{12}$$

One can see that the magnitude of each matrix element of ΔU is of $\mathcal{O}(0.1)$, implying that the realistic pattern of U might result from a democratic perturbation to U_0 (i.e., the nine entries of ΔU are all proportional to a common small parameter). We shall elaborate on this point in detail in section VII.

(2) The bimaximal mixing pattern of lepton flavors [12]:

$$U_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{13}$$

which has $\theta_{12}^{(0)} = 45^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = 45^\circ$ in the standard parametrization. Comparing Eq. (13) with Eq. (10), we obtain the perturbation matrix

$$\Delta U = \begin{pmatrix} 0.112 & -0.155 & 0.156 e^{-i\delta} \\ 0.105 - 0.092 e^{i\delta} & 0.086 - 0.062 e^{i\delta} & -0.009 \\ -0.105 - 0.092 e^{i\delta} & -0.086 - 0.062 e^{i\delta} & -0.009 \end{pmatrix} . \tag{14}$$

We see that the matrix elements $(\Delta U)_{\mu 3}$ and $(\Delta U)_{\tau 3}$ are highly suppressed. In other words, the initially maximal angle $\theta_{23}^{(0)}$ receives the minimal correction, which is much smaller than the one received by the initially minimal angle $\theta_{13}^{(0)}$. Such a situation seems to be more or less unnatural, at least from a point of view of model building.

(3) The tri-bimaximal mixing pattern of lepton flavors [13]:

$$U_0 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{15}$$

whose three mixing angles are $\theta_{12}^{(0)} = \arctan(1/\sqrt{2}) \simeq 35.3^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = 45^\circ$ in the standard parametrization. In a similar way we get the corresponding perturbation matrix

$$\Delta U = \begin{pmatrix} 0.003 & -0.025 & 0.156 e^{-i\delta} \\ 0.013 - 0.092 e^{i\delta} & 0.009 - 0.062 e^{i\delta} & -0.009 \\ -0.013 - 0.092 e^{i\delta} & -0.009 - 0.062 e^{i\delta} & -0.009 \end{pmatrix} . \tag{16}$$

It is quite obvious that $(\Delta U)_{e1}$, $(\Delta U)_{e2}$, $(\Delta U)_{\mu 3}$ and $(\Delta U)_{\tau 3}$ are highly suppressed. So two initially large angles $\theta_{12}^{(0)}$ and $\theta_{23}^{(0)}$ are only slightly modified by the perturbation effects, but the initially minimal angle $\theta_{13}^{(0)}$ receives the maximal correction.

(4) The golden-ratio mixing pattern of lepton flavors [14]:

$$U_0 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} & \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} & 0 \\ -\frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5+\sqrt{5}}} & -\frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{17}$$

which has $\theta_{12}^{(0)} = \arctan[2/(1+\sqrt{5})] \simeq 31.7^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = 45^\circ$ in the standard parametrization. In this case the perturbation matrix ΔU turns out to be

$$\Delta U = \begin{pmatrix} -0.032 & 0.026 & 0.156 e^{-i\delta} \\ -0.023 - 0.092 e^{i\delta} & -0.016 - 0.062 e^{i\delta} & -0.009 \\ 0.023 - 0.092 e^{i\delta} & 0.016 - 0.062 e^{i\delta} & -0.009 \end{pmatrix} . \tag{18}$$

Similar to the tri-bimaximal mixing pattern, two initially large angles of the golden-ratio mixing pattern are only slightly corrected, but the initially minimal angle $\theta_{13}^{(0)}$ is significantly modified by the same perturbation.

(5) The hexagonal mixing pattern of lepton flavors [15]:

$$U_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{19}$$

whose mixing angles are $\theta_{12}^{(0)} = 30^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = 45^\circ$ in the standard parametrization. In this case we obtain the perturbation matrix

$$\Delta U = \begin{pmatrix} -0.047 & 0.052 & 0.156 e^{-i\delta} \\ -0.041 - 0.092 e^{i\delta} & -0.026 - 0.062 e^{i\delta} & -0.009 \\ 0.041 - 0.092 e^{i\delta} & 0.026 - 0.062 e^{i\delta} & -0.009 \end{pmatrix} . \tag{20}$$

This result is quite analogous to the one obtained in Eq. (16) or Eq. (18), simply because the patterns of U_0 in these three cases are quite similar.

Now let us summarize some useful lessons that we can directly learn from the above five typical examples of U.

- To accommodate the unsuppressed value of θ_{13} in a generic flavor mixing structure $U = (U_0 + \Delta U)\, P_\nu$, one has to choose a proper constant mixing pattern U_0 and adjust its perturbation matrix ΔU. The phenomenological criterion to do so is two-fold: on the one hand, U_0 should easily be derived from a certain flavor symmetry; on the other hand, ΔU should have a natural structure which can easily be accounted for by either the flavor symmetry breaking or quantum corrections (or both of them).

- The common feature of the above five patterns of U_0 is apparently $(U_0)_{e3} = 0$ (or equivalently, $\theta_{13}^{(0)} = 0$), implying that a relatively large perturbation is required for generating $\theta_{13} \sim 9^\circ$. In this case, the closer $\theta_{12}^{(0)}$ and $\theta_{23}^{(0)}$ are to the observed values of θ_{12} and θ_{23}, the more unnatural the structure of ΔU seems to be. The tri-bimaximal mixing pattern given in Eq. (15), which is currently the most popular ansatz for model building based on certain flavor symmetries, suffers from this unnaturalness in particular [26]. In this sense we argue that the democratic mixing pattern in Eq. (11) might be more natural and deserve some more attention.

- One may certainly consider some possible patterns of U_0 which can predict a finite value of $\theta_{13}^{(0)}$ in the vicinity of the experimental value of θ_{13}. In this case the three mixing angles of U_0 may receive comparably small corrections from the perturbation matrix ΔU, and thus the naturalness criterion can be satisfied. For example, the following two patterns of U_0 belong to this category and have been discussed in the literature [1]: one of them is the so-called correlative mixing pattern [26] [2]

$$U_0 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} c_* & \frac{1}{\sqrt{3}} c_* & s_* e^{-i\delta} \\ -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} s_* e^{i\delta} & \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} s_* e^{i\delta} & \frac{1}{\sqrt{2}} c_* \\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} s_* e^{i\delta} & -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} s_* e^{i\delta} & \frac{1}{\sqrt{2}} c_* \end{pmatrix} \tag{21}$$

  with $c_* \equiv \cos\theta_* = (\sqrt{2}+1)/\sqrt{6}$ and $s_* \equiv \sin\theta_* = (\sqrt{2}-1)/\sqrt{6}$, which predicts $\theta_{12}^{(0)} = \arctan(1/\sqrt{2}) \simeq 35.3^\circ$, $\theta_{23}^{(0)} = 45^\circ$ and $\theta_{13}^{(0)} = \theta_{23}^{(0)} - \theta_{12}^{(0)} \simeq 9.7^\circ$; and the other is the tetra-maximal mixing pattern [28]

---

[1] A more detailed analysis of possible forms of U_0 has been done in Ref. [27].

[2] The reason for this name is simply that the three flavor mixing angles in this constant pattern exactly satisfy the interesting correlative relation $\theta_{12}^{(0)} + \theta_{13}^{(0)} = \theta_{23}^{(0)}$.

$$U_0 = \begin{pmatrix} \frac{2+\sqrt{2}}{4} & \frac{1}{2} & \frac{2-\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} + \frac{i\left(\sqrt{2}-1\right)}{4} & \frac{1}{2} - \frac{i\sqrt{2}}{4} & \frac{\sqrt{2}}{4} + \frac{i\left(\sqrt{2}+1\right)}{4} \\ -\frac{\sqrt{2}}{4} - \frac{i\left(\sqrt{2}-1\right)}{4} & \frac{1}{2} + \frac{i\sqrt{2}}{4} & \frac{\sqrt{2}}{4} - \frac{i\left(\sqrt{2}+1\right)}{4} \end{pmatrix} \tag{22}$$

which predicts $\theta_{12}^{(0)} = \arctan(2-\sqrt{2}) \simeq 30.4^\circ$, $\theta_{23}^{(0)} = 45^\circ$ and $\theta_{13}^{(0)} = \arcsin[(2-\sqrt{2})/4] \simeq 8.4^\circ$. Of course, whether such constant mixing patterns can easily be derived from some underlying flavor symmetries remains an open question.

In short, today's model building has to take the challenge caused by the reasonably large value of θ_{13} as observed in the Daya Bay experiment [9].

Furthermore, it is worth mentioning that the renormalization-group running effects or finite quantum corrections are almost impossible to generate $\theta_{13} \simeq 9^\circ$ from $\theta_{13}^{(0)} = 0^\circ$, unless the seesaw threshold effects or other extreme conditions are taken into account [29]. One may therefore consider a pattern of U_0 with nonzero $\theta_{13}^{(0)}$, such as the tetra-maximal mixing pattern [30] or the correlative mixing pattern [31], as a starting point of view to calculate the radiative corrections before confronting it with current experimental data. We shall elaborate on this idea and examine its impact on leptonic CP violation elsewhere [31].

IV. THE MINIMAL PERTURBATION TO U_0

Note that the perturbation matrix ΔU in Eq. (4) is in general a sum of all possible perturbations to the constant flavor mixing matrix U_0. From the point of view of model building, it is helpful to single out a viable ΔU whose form is as simple as possible. To do so, let us reexpress Eq. (4) in the following manner:

$$U = \left(U_0 + \Delta U\right) P_\nu = U_0 \left(\mathbf{1} + \Delta U'\right) P_\nu = \left(\mathbf{1} + \Delta U'_{\rm L}\right) U_0 \left(\mathbf{1} + \Delta U'_{\rm R}\right) P_\nu \; , \tag{23}$$

where $\Delta U = U_0 \Delta U' = \Delta U'_{\rm L} U_0 + U_0 \Delta U'_{\rm R} + \Delta U'_{\rm L} U_0 \Delta U'_{\rm R}$ holds, and it satisfies the condition $U_0 \Delta U^\dagger + \Delta U U_0^\dagger + \Delta U \Delta U^\dagger = \mathbf{0}$ as a result of the unitarity of U itself. Therefore, one may achieve a viable but minimal perturbation to U_0 by switching off $\Delta U'_{\rm L}$ (or $\Delta U'_{\rm R}$) and adjusting $\Delta U'_{\rm R}$ (or $\Delta U'_{\rm L}$) to its simplest form which is allowed by current experimental data. Such a treatment is actually equivalent to multiply U_0 by a unitary perturbation matrix, which may more or less deviate from the identity matrix $\mathbf{1}$, from either its left-hand side or its right-hand side. The first example of this kind was given in Ref. [10] for the democratic mixing pattern, and its ΔU was mainly responsible for the generation of nonzero θ_{13} and δ.

Here we concentrate on the typical patterns of U_0 discussed above and outline the main ideas of choosing the minimal perturbations to them.

- If U_0 predicts $\theta_{23}^{(0)} = 45^\circ$ and $\theta_{13}^{(0)} = 0^\circ$ together with $\theta_{12}^{(0)} > 34^\circ$ (the best-fit value based on current neutrino oscillation data [3]), then the simplest way to generate a relatively large θ_{13}, keep $\theta_{23} = \theta_{23}^{(0)} = 45^\circ$ unchanged and correct $\theta_{12}^{(0)}$ to a slightly smaller value is to choose a complex $(2,3)$ rotation matrix as the perturbation matrix:

$$\mathbf{1} + \Delta U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & i\sin\theta \\ 0 & i\sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \Delta U' \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sin^2\theta & i\sin\theta \\ 0 & i\sin\theta & -\frac{1}{2}\sin^2\theta \end{pmatrix} \; , \tag{24}$$

where θ is a small angle to trigger the perturbation effect. The most striking example in this category is to take U_0 to be the tri-bimaximal mixing pattern given in Eq. (15). The result is [32]:

$$U = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}}\cos\theta & \frac{i}{\sqrt{3}}\sin\theta \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{\sqrt{2}}\cos\theta + \frac{i}{\sqrt{3}}\sin\theta \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{\sqrt{2}}\cos\theta - \frac{i}{\sqrt{3}}\sin\theta \end{pmatrix} P_\nu \; , \tag{25}$$

which predicts

$$\sin^2\theta_{12} = \frac{1}{3}\left(1 - 2\tan^2\theta_{13}\right) \; , \quad \sin^2\theta_{13} = \frac{1}{3}\sin^2\theta \; , \quad \theta_{23} = 45^\circ \; , \quad \delta = 90^\circ \tag{26}$$

in the standard parametrization. Note that the obtained correlation between θ_{12} and θ_{13} is especially interesting because it leads us to $\theta_{12} \to 34^\circ$ when $\theta_{13} \to 9^\circ$, consistent with the present experimental data. If θ_{23} is allowed to slightly deviate from $\theta_{23}^{(0)} = 45^\circ$, then one may simply make the replacement $i \to e^{i\delta}$ in Eq. (25).

- If U_0 predicts $\theta_{23}^{(0)} = 45^\circ$ and $\theta_{13}^{(0)} = 0^\circ$ together with $\theta_{12}^{(0)} < 34^\circ$, then the most economical way to generate a relatively large θ_{13}, keep $\theta_{23} = \theta_{23}^{(0)} = 45^\circ$ unchanged and correct $\theta_{12}^{(0)}$ to a slightly larger value is to choose a complex $(1,3)$ rotation matrix as the perturbation matrix:

$$\mathbf{1} + \Delta U' = \begin{pmatrix} \cos\theta & 0 & i\sin\theta \\ 0 & 1 & 0 \\ i\sin\theta & 0 & \cos\theta \end{pmatrix} \quad \text{or} \quad \Delta U' \simeq \begin{pmatrix} -\frac{1}{2}\sin^2\theta & 0 & i\sin\theta \\ 0 & 0 & 0 \\ i\sin\theta & 0 & -\frac{1}{2}\sin^2\theta \end{pmatrix} \; . \tag{27}$$

Taking U_0 to be the golden-ratio mixing pattern given in Eq. (17) for example [3], we immediately arrive at

$$U = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}}\cos\theta & \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} & \frac{i\sqrt{2}}{\sqrt{5-\sqrt{5}}}\sin\theta \\ -\frac{1}{\sqrt{5+\sqrt{5}}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{2}}\cos\theta - \frac{i}{\sqrt{5+\sqrt{5}}}\sin\theta \\ \frac{1}{\sqrt{5+\sqrt{5}}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta & -\frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{2}}\cos\theta + \frac{i}{\sqrt{5+\sqrt{5}}}\sin\theta \end{pmatrix} P_\nu \; , \tag{28}$$

whose predictions include $\theta_{23} = 45^\circ$, $\delta = 90^\circ$, and

$$\sin^2\theta_{12} = \frac{2}{5+\sqrt{5}}\left(1 + \tan^2\theta_{13}\right) \; , \qquad \sin^2\theta_{13} = \frac{2}{5-\sqrt{5}}\sin^2\theta \tag{29}$$

in the standard parametrization of U. In this case the correlation between θ_{12} and θ_{13} leads us to $\theta_{12} \to 32^\circ$ when $\theta_{13} \to 9^\circ$, compatible with current experimental data. Again, the replacement $i \to e^{i\delta}$ in Eq. (28) allows one to obtain a somewhat more flexible value of θ_{23} which may slightly deviate from $\theta_{23}^{(0)} = 45^\circ$.

---

[3]An interesting example with U_0 being the tri-bimaximal mixing pattern has been discussed in Ref. [33], but this ansatz predicts θ_{12} to be slightly larger than $\theta_{12}^{(0)} \simeq 35.3^\circ$.

- If U_0 is quite far away from the realistic MNSP matrix U, one has to consider a somewhat complicated perturbation matrix including two rotation angles. In the neglect of CP violation, for instance, we may consider

$$\mathbf{1} + \Delta U' = \begin{pmatrix} c'_{12} & -s'_{12} & 0 \\ s'_{12}c'_{23} & c'_{12}c'_{23} & s'_{23} \\ s'_{12}s'_{23} & c'_{12}s'_{23} & -c'_{23} \end{pmatrix} , \tag{30}$$

  where $c'_{ij} \equiv \cos\theta'_{ij}$ and $s'_{ij} \equiv \sin\theta'_{ij}$ (for $ij = 12, 23$). However, we hope that the resulting structure of U still allows us to obtain one or two predictions, in particular for the mixing angle θ_{13}. An example of this kind has been given in Ref. [34] by taking U_0 to be the democratic mixing pattern, and it predicts an interesting relationship between θ_{13} and θ_{23} in the standard parametrization:

$$\sin\theta_{13} = \frac{\sqrt{2} - \tan\theta_{23}}{\sqrt{5 - 2\sqrt{2}\tan\theta_{23} + 4\tan^2\theta_{23}}} . \tag{31}$$

  Typically taking $\theta_{23} \simeq 45^\circ$, we arrive at $\theta_{13} \simeq 9.6^\circ$ [34], which is in agreement with the Daya Bay result [9]. It is easy to accommodate a CP-violating phase in $\Delta U'$ [34], although its form might not be really minimal anymore.

For those constant flavor mixing patterns with $\theta_{13}^{(0)} \neq 0^\circ$ from the very beginning, such as the correlative [26] and tetra-maximal [28] mixing scenarios given in Eqs. (21) and (22), the similar minimal perturbations can be introduced in order to make the resulting MNSP matrix U fit the experimental data to a much better degree of accuracy.

It should be noted that the above discussions about possible patterns of ΔU (or $\Delta U'$) with respect to those of U_0 are purely phenomenological. From the point of view of model building, it is more meaningful to consider the textures of lepton mass matrices

$$M_l = M_l^{(0)} + \Delta M_l , \qquad\qquad M_\nu = M_\nu^{(0)} + \Delta M_\nu , \tag{32}$$

where $M_l^{(0)}$ and $M_\nu^{(0)}$ can be obtained in the limit of certain flavor symmetries, and their special structures allow us to achieve a constant flavor mixing pattern U_0. The perturbation matrices ΔM_l and ΔM_ν play an important role in transforming U_0 into the realistic MNSP matrix U, and thus their textures should be determined in a simple way and with a good reason. The connection between $\Delta M_{l,\nu}$ and ΔU (or $\Delta U'$) depends on the details of a lepton flavor model and may not be very transparent in most cases. In the basis where M_l is real and positive, however, ΔM_ν can be formally expressed as

$$\Delta M_\nu = \left(U_0 + \Delta U\right) \overline{M}_\nu \left(U_0 + \Delta U\right)^T - U_0 \overline{M}_\nu^{(0)} U_0^T , \tag{33}$$

in which $\overline{M}_\nu = P_\nu \widehat{M}_\nu P_\nu^T$ and $\overline{M}_\nu^{(0)} = P'_\nu \widehat{M}'_\nu P'^T_\nu$ together with $\widehat{M}'_\nu \equiv \mathrm{Diag}\{m'_1, m'_2, m'_3\}$ and $P'_\nu \equiv \mathrm{Diag}\{e^{i\rho'}, e^{i\sigma'}, 1\}$. Here m'_i (for $i = 1, 2, 3$) denote the eigenvalues of $M_\nu^{(0)}$ in the symmetry limit, while ρ' and σ' stand for the Majorana phases in the same limit. It is therefore possible, at least in principle, to fix the structure of ΔM_ν with the help of a certain flavor symmetry and current experimental data.

V. ON μ-τ SYMMETRY AND CP VIOLATION

Let us proceed to discuss two other flavor issues in the lepton sector after the successful measurement of the smallest mixing angle θ_{13} [9]. One of them is about a possible departure of the largest mixing angle θ_{23} from 45°, and the other is about the strength of leptonic CP violation. The former is an important issue in neutrino phenomenology, because it crucially determines the structure of the MNSP matrix U; and the latter is certainly more important because the observed matter-antimatter asymmetry of the Universe might be associated with leptonic CP violation at low energies via the seesaw and leptogenesis mechanisms [35].

It is well known that $\theta_{23} \simeq 45°$ is favored by current atmospheric and accelerator neutrino oscillation data [1]. If θ_{23} is exactly equal to 45°, then one may arrive at a partial μ-τ permutation symmetry in the MNSP matrix U (i.e., the equality $|U_{\mu 3}| = |U_{\tau 3}|$). This point can easily be seen from Eq. (10), where $\theta_{23} \simeq 45°$ has typically been input. The full μ-τ symmetry of U in modulus is described by the equalities

$$|U_{\mu 1}| = |U_{\tau 1}| \; , \qquad |U_{\mu 2}| = |U_{\tau 2}| \; , \qquad |U_{\mu 3}| = |U_{\tau 3}| \; , \tag{34}$$

equivalent to two independent sets of conditions in the standard parametrization [36]:

$$\theta_{23} = 45° \; , \quad \theta_{13} = 0° \; , \tag{35}$$

or

$$\theta_{23} = 45° \; , \quad \delta = \pm 90° \; . \tag{36}$$

One can see that the constant mixing patterns in Eqs. (13), (15), (17) and (19) satisfy the conditions in Eq. (35), while those in Eqs. (22), (25) and (28) satisfy the conditions in Eq. (36) [4]. Hence these seven scenarios of the MNSP matrix U all have the complete μ-τ symmetry in modulus, or equivalently the equalities $|U_{\mu i}| = |U_{\tau i}|$ (for $i = 1, 2, 3$). Now that $\theta_{13} \neq 0°$ has firmly been established by the Daya Bay experiment [9], we are therefore concerned about a possible deviation of θ_{23} from 45° and (or) a possible departure of δ from $\pm 90°$. We speculate that U might have an approximate μ-τ symmetry with $|U_{\mu i}| \simeq |U_{\tau i}|$, in contrast with the approximate off-diagonal symmetry of the CKM matrix V in modulus (i.e., $|V_{us}| \simeq |V_{cd}|$, $|V_{cb}| \simeq |V_{ts}|$ and $|V_{ub}| \simeq |V_{td}|$ [1]).

In the basis where the flavor eigenstates of three charged leptons are identified with their mass eigenstates (i.e., $M_l = \widehat{M}_l$), the Majorana neutrino mass matrix of the form

$$M_\nu = \begin{pmatrix} a & b & -b \\ b & c & d \\ -b & d & c \end{pmatrix} \tag{37}$$

predicts the μ-τ permutation symmetry of the MNSP matrix U with $\theta_{13} = 0°$ and $\theta_{23} = 45°$; while the mass matrix of the form

---

[4]The correlative mixing pattern in Eq. (21) may also satisfy the conditions in Eq. (36) if its CP-violating phase δ is taken to be $\pm 90°$.

$$
M_\nu = \begin{pmatrix} a & b & -b^* \\ b & c & d \\ -b^* & d & c^* \end{pmatrix} \tag{38}
$$

leads us to the the μ-τ symmetry of U with $\delta = \pm 90^\circ$ and $\theta_{23} = 45^\circ$. In either of the above textures of M_ν, its entries have certain kinds of linear correlations or equalities and thus can be generated from some underlying flavor symmetries. In view of the experimental evidence for $\theta_{13} \neq 0^\circ$ [9], the pattern of M_ν in Eq. (37) has to be modified. For a similar reason, the more reliable and accurate experimental knowledge on θ_{23} and δ will be extremely useful for us to identify the effect of μ-τ symmetry breaking and build more realistic models for lepton mass generation, flavor mixing and CP violation.

The fact that θ_{13} is not strongly suppressed is certainly a good news to the experimental attempts towards a measurement of CP violation in the lepton sector. The strength of CP violation in neutrino oscillations is described by the Jarlskog rephasing invariant [37]

$$
\mathcal{J}_l = \mathrm{Im}\left(U_{e1} U_{\mu 2} U^*_{e2} U^*_{\mu 1}\right) = \mathrm{Im}\left(U_{e2} U_{\mu 3} U^*_{e3} U^*_{\mu 2}\right) = \cdots = c_{12} s_{12} c^2_{13} s_{13} c_{23} s_{23} \sin\delta \ , \tag{39}
$$

which is proportional to the sine of the smallest flavor mixing angle θ_{13}. In the quark sector one has determined the corresponding Jarlskog invariant $\mathcal{J}_q \simeq 3 \times 10^{-5}$ [1] and attributed its smallness to the strongly suppressed values of quark flavor mixing angles (i.e., $\vartheta_{\rm C} \equiv \vartheta_{12} \simeq 13^\circ$, $\vartheta_{13} \simeq 0.2^\circ$ and $\vartheta_{23} \simeq 2.4^\circ$). In the lepton sector both θ_{12} and θ_{23} are large, and thus it is possible to achieve a relatively large value of $\mathcal{J}_l$ if the CP-violating phase δ is not suppressed either. Note that the maximal value of $\mathcal{J}_l$ or $\mathcal{J}_q$ can be obtained only when the MNSP (or CKM) matrix takes the special Cabibbo texture $V_{\rm C}$ [38] or its equivalent form $V'_{\rm C}$ in the standard-parametrization phase convention:

$$
V_{\rm C} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \end{pmatrix} \quad \Longrightarrow \quad V'_{\rm C} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ -\frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right) & \frac{1}{2}\left(1 - \frac{i}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}} \\ \frac{1}{2}\left(1 - \frac{i}{\sqrt{3}}\right) & -\frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}} \end{pmatrix} \ , \tag{40}
$$

where $\omega = e^{i2\pi/3}$ is the complex cube-root of unity (i.e., $\omega^3 = 1$). Therefore, $V_{\rm C}$ or $V'_{\rm C}$ predicts $\theta_{12} = \theta_{23} = 45^\circ$, $\theta_{13} = \arctan(1/\sqrt{2}) \simeq 35.3^\circ$ and $\delta = 90^\circ$, leading to the maximal CP violation $\mathcal{J}_{\rm max} = 1/(6\sqrt{3}) \simeq 9.6 \times 10^{-2}$. Unfortunately, both the CKM matrix V and the MNSP matrix U are remarkably different from the Cabibbo matrix $V_{\rm C}$. We see $\mathcal{J}_q/\mathcal{J}_{\rm max} \simeq 3 \times 10^{-4}$, and hence CP violation is rather weak in the quark sector. Taking $\theta_{12} \simeq 34^\circ$, $\theta_{13} \sim 9^\circ$ and $\theta_{23} \simeq 45^\circ$ as a realistic example of U, we arrive at $\mathcal{J}_l/\mathcal{J}_{\rm max} \simeq 0.37 \sin\delta$, implying that the magnitude of leptonic CP violation can actually reach the percent level in neutrino oscillations if the CP-violating phase δ is not strongly suppressed (e.g., $\delta \gtrsim 16^\circ$ for the values of three mixing angles taken above). Whether CP violation is significant or not turns out to be an important question in lepton physics, especially in neutrino phenomenology.

Note that $\mathcal{J}_l \neq 0$ is a necessary and sufficient condition for leptonic CP violation. In particular, the determinant of the commutator of lepton mass matrices [41]

$$
\begin{aligned}
& {\rm Det}\left(i\left[M_\nu M^\dagger_\nu, M_l M^\dagger_l\right]\right) \\
= & \; 2\mathcal{J}_l \left(m^2_e - m^2_\mu\right)\left(m^2_\mu - m^2_\tau\right)\left(m^2_\tau - m^2_e\right)\left(m^2_1 - m^2_2\right)\left(m^2_2 - m^2_3\right)\left(m^2_3 - m^2_1\right)
\end{aligned} \tag{41}
$$

is unable to provide us with any more information about CP violation. The reason is simply that $\mathcal{J}_l$ would automatically vanish if the masses of two charged leptons or two neutrinos became degenerate [42]. In other words, one may consider the conditions for CP violation either at the level of lepton flavor mixing (i.e., $\mathcal{J}_l$ or δ) or at the level of lepton mass matrices, but a confusion or double-counting problem may occur if the conditions obtained at two different levels are mixed like Eq. (41). The same observation is true in the quark sector, as already pointed out in Ref. [43].

A geometrical description of CP violation in terms of the unitarity triangles has proved very useful in the quark sector [1]. This language was first applied to the lepton sector in Ref. [16], in which six leptonic unitarity triangles have been named as $\triangle_e$, $\triangle_\mu$, $\triangle_\tau$ and $\triangle_1$, $\triangle_2$, $\triangle_3$ (see FIG. 1 for illustration). They totally have nine independent inner angles and eighteen independent sides, but their areas are all equal to $\mathcal{J}_l/2$ as dictated by the unitarity of U itself [5]. If $U = V_{\rm C}$ is taken, then the six unitarity triangles are congruent with one another and converge to an equilateral triangle whose sides are all equal to 1/3 and whose area is equal to $\mathcal{J}_{\rm max}/2$. The fact that U is rather different from $V_{\rm C}$ means somewhat smaller CP-violating effects in the leptonic charged-current interactions. Given $\delta \simeq 90^\circ$ together with $\theta_{12} \simeq 34^\circ$, $\theta_{13} \sim 9^\circ$ and $\theta_{23} \simeq 45^\circ$, for instance, the nine inner angles of the six unitarity triangles in FIG. 1 turn out to be

$$\Phi \equiv \begin{pmatrix} \Phi_{e1} & \Phi_{e2} & \Phi_{e3} \\ \Phi_{\mu 1} & \Phi_{\mu 2} & \Phi_{\mu 3} \\ \Phi_{\tau 1} & \Phi_{\tau 2} & \Phi_{\tau 3} \end{pmatrix} \simeq \begin{pmatrix} 12.05^\circ & 26.11^\circ & 141.8^\circ \\ 83.98^\circ & 76.94^\circ & 19.08^\circ \\ 83.98^\circ & 76.94^\circ & 19.08^\circ \end{pmatrix} . \tag{42}$$

We see that this unitarity-triangle angle matrix exhibits an interesting μ-τ symmetry as guaranteed by the inputs $\delta \simeq 90^\circ$ and $\theta_{23} \simeq 45^\circ$. In addition, its nine matrix elements are rephasing-invariant and satisfy the sum rules [44]

$$\sum_\alpha \Phi_{\alpha i} = \sum_i \Phi_{\alpha i} = 180^\circ \; , \tag{43}$$

where the subscript α runs over e, μ and τ, and i runs over 1, 2 and 3. We expect that the future long-baseline neutrino oscillation experiments can hopefully determine or constrain some of the above angles and thus pin down the CP-violating phase δ of U even in the presence of terrestrial matter effects on the unitarity triangles [45].

VI. NINE DISTINCT PARAMETRIZATIONS

The 3×3 unitary flavor mixing matrix can always be parametrized in terms of three rotation angles and a few phase angles. A classification of all the possible parametrizations of this kind has been done in Ref. [46]. Here we list nine topologically distinct parametrizations of the MNSP matrix U in TABLE I, in which three rotation matrices are defined as

---

[5] If the unitarity of U is directly violated in the presence of light sterile neutrinos or indirectly broken due to the existence of heavy sterile neutrinos, such unitarity triangles will change to the quadrangles [39] or polygons [40] in which new CP-violating effects must be included.

$$
\begin{aligned}
R_{12}(\theta_{12}, \delta) &= \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & e^{-i\delta} \end{pmatrix} , \\
R_{23}(\theta_{23}, \delta) &= \begin{pmatrix} e^{-i\delta} & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} , \\
R_{13}(\theta_{13}, \delta) &= \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & e^{-i\delta} & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} ,
\end{aligned} \tag{44}
$$

with $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ (for $ij = 12, 13, 23$). Although all the parametrizations of U (or the CKM matrix V) are mathematically equivalent, we argue that some of them might be phenomenologically more interesting in the sense that they might either make the underlying dynamics of flavor mixing more transparent or lead to more straightforward and simpler relations between fundamental parameters and observable quantities [46]. In other words, they are possible to provide us with some novel points of view on the structure of lepton or quark flavor mixing. As stressed by Feynman, "different views suggest different kinds of modifications which might be made" and "a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory available to him" [47].

Let us focus on the MNSP matrix U and make some comments on its nine different parametrizations listed in TABLE I.

- Pattern (1) was first proposed in Ref. [48] , and it is usually expressed in terms of the following notations:

$$
U = \begin{pmatrix} s_l s_\nu c + c_l c_\nu e^{-i\varphi} & s_l c_\nu c - c_l s_\nu e^{-i\varphi} & s_l s \\ c_l s_\nu c - s_l c_\nu e^{-i\varphi} & c_l c_\nu c + s_l s_\nu e^{-i\varphi} & c_l s \\ -s_\nu s & -c_\nu s & c \end{pmatrix} P_\nu , \tag{45}
$$

  where $c_{l,\nu} \equiv \cos\theta_{l,\nu}$, $s_{l,\nu} \equiv \sin\theta_{l,\nu}$, $c \equiv \cos\theta$ and $s \equiv \sin\theta$. In the leading-order approximation we have $s_\nu \simeq s_{12}$, $s \simeq s_{23}$ and $s_l \simeq s_{13}/s_{23}$. There are two remarkable merits of this parametrization: 1) it is quite useful for model building if the neutrino mass spectrum has a normal hierarchy as the charged-lepton or quark mass spectrum (e.g., $\tan\theta_l \simeq \sqrt{m_e/m_\mu}$ and $\tan\theta_\nu \simeq \sqrt{m_1/m_2}$ have been conjectured in Ref. [18]); and 2) it allows us to obtain impressively simple expressions of the one-loop renormalization-group equations for three flavor mixing angles and three CP-violating phases, much simpler than those obtained by using the standard parametrization in Eq. (2) [49].

- Pattern (2) is equivalent to the original Kobayashi-Maskawa parametrization [4]. The structure of this pattern and those of patterns (3) and (7) have a common feature: the rotation matrix on the left-hand side of U is $R_{23}(\theta_{23})$, which is commutable with a diagonal matrix of the form Diag$\{A, 0, 0\}$. When a neutrino beam travels through a normal medium, the coherent forward scattering effect induced by the charged-current interactions of electron neutrinos (or antineutrinos) with matter can just generate such an effective potential term [50]. Hence patterns (2), (3) and (7) are more convenient to describe matter effects on neutrino oscillations. In particular, it has been shown in Ref. [51] that these three parametrizations of U can all lead us to the exact and interesting Toshev relation [52]

$$\sin\tilde{\delta}\sin 2\tilde{\theta}_{23} = \sin\delta\sin 2\theta_{23} \; , \tag{46}$$

where $\tilde{\theta}_{23}$ and $\tilde{\delta}$ denote the effective counterparts of θ_{23} and δ in matter.

- Pattern (3) is equivalent to the standard parametrization of U given in Eq. (2) [1], although its phase convention is slightly different. This representation becomes most popular today because its three mixing angles (θ_{12}, θ_{23}, θ_{13}) directly measure the effects of solar, atmospheric and reactor neutrino oscillations ($\sin^2 2\theta_{12}$, $\sin^2 2\theta_{23}$, $\sin^2 2\theta_{13}$) in the two-flavor approximation in vacuum. Furthermore, the smallest mixing angle θ_{13} determines the smallest matrix element U_{e3} of the MNSP matrix U in a way analogous to the standard parametrization of the CKM matrix V, where the smallest element V_{ub} is controlled by the smallest mixing angle ϑ_{13} [1]. Hence in this parametrization the hierarchy of three mixing angles can almost truly reflect the overall hierarchy of the flavor mixing matrix, as we have discussed in sections III, IV and V.

- Pattern (5) is structurally special in the sense that only the 3×3 flavor mixing matrix U can have this form around its "central element" $U_{\mu 2}$. As a result, two off-diagonal asymmetries of U in modulus can simply be expressed as

$$\begin{aligned} \mathcal{A}_{\rm L} &\equiv |U_{e2}|^2 - |U_{\mu 1}|^2 = |U_{\mu 3}|^2 - |U_{\tau 2}|^2 = |U_{\tau 1}|^2 - |U_{e3}|^2 = s^2_{12}\left(c^2_{13} - c^{\prime 2}_{13}\right) \; , \\ \mathcal{A}_{\rm R} &\equiv |U_{e2}|^2 - |U_{\mu 3}|^2 = |U_{\mu 1}|^2 - |U_{\tau 2}|^2 = |U_{\tau 3}|^2 - |U_{e1}|^2 = s^2_{12}\left(c^2_{13} - s^{\prime 2}_{13}\right) \; . \end{aligned} \tag{47}$$

  Current neutrino oscillation data indicate that both $\mathcal{A}_{\rm L} \neq 0$ and $\mathcal{A}_{\rm R} \neq 0$ hold at the 3σ level, implying that the MNSP matrix U is apparently asymmetric in modulus about either its U_{e1}-$U_{\mu 2}$-$U_{\tau 3}$ axis or its U_{e3}-$U_{\mu 2}$-$U_{\tau 1}$ axis [53]. In contrast, the CKM matrix V is roughly symmetric in modulus about its V_{ud}-V_{cs}-V_{tb} axis. Another unique feature of pattern (5) is that it assures three mixing angles to be comparably large and the (Dirac) CP-violating phase to be nearly minimal (in particular, $\vartheta_{12} \simeq 13.2^\circ$, $\vartheta_{13} \simeq 10.1^\circ$, $\vartheta^\prime_{13} \simeq 10.3^\circ$ and $\delta \simeq 1.1^\circ$ for the quark sector; and all the three mixing angles are around 45° for the lepton sector with a much smaller Dirac CP-violating phase [54]). In this sense the approximate flavor mixing democracy and minimal CP violation have been discussed in Ref. [54] as a different point of view to look at the flavor puzzles of leptons and quarks.

- Some interest has also been paid to patterns (4), (6) and (8) [55] for two simple reasons: 1) none of the three flavor mixing angles is suppressed in each of them; and 2) the CP-violating phase δ is strongly correlated with the mixing angles. This kind of strong parameter correlation might allow one to determine δ with fewer uncertainties from an experimental point of view, as compared with the relatively weak parameter correlation in patterns (3), (7) and (9), where the value of θ_{13} is much smaller than those of θ_{12} and θ_{23}. Generally speaking, however, patterns (4), (6), (7), (8) and (9) seem to be somewhat less interesting than patterns (1), (2), (3) and (5) for the phenomenological studies of flavor physics.

For each of the nine parametrizations of the MNSP matrix U, the explicit expression of the Jarlskog invariant of leptonic CP violation $\mathcal{J}_l$ has been given in TABLE I.

VII. THE WOLFENSTEIN-LIKE EXPANSION

Following the conjecture that the MNSP matrix U is composed of a constant leading term U_0 and a perturbation term ΔU as described in Eq. (4), we have argued that the structure of ΔU is relatively natural if U_0 takes the democratic mixing pattern. In particular, the numerical result of ΔU obtained in Eq. (12) indicates that its nine matrix elements are all of $\mathcal{O}(0.1)$ and thus can easily be described by a common small parameter. This observation reminds us of the well-known Wolfenstein parametrization of the CKM matrix V, which was proposed soon after the smallest element V_{ub} was experimentally determined [11]. Such a parametrization has proved to be very useful because it clearly reveals the observed strong hierarchy in the quark flavor structure. Although a straightforward Wolfenstein-like parametrization of the MNSP matrix U has been discussed [15], it is not useful because the structure of U is not as hierarchical as that of V. A different starting point of view is to speculate that the realistic form of U comes from the democratic mixing pattern U_0 and a Wolfenstein-like perturbation ΔU. Here we proceed to explore this noteworthy possibility in some detail, so as to illustrate an alternative way for describing the phenomenon of lepton flavor mixing other than those parametrizations discussed in section VI.

Comparing Eq. (11) with Eq. (2), we can define three Wolfenstein-like parameters in the following way:

$$\begin{aligned} \theta_{12} &\equiv \theta_{12}^{(0)} - \theta_x \quad \text{with} \quad \sin\theta_x \equiv \lambda \ , \\ \theta_{23} &\equiv \theta_{23}^{(0)} - \theta_y \quad \text{with} \quad \sin\theta_y \equiv A\lambda \ , \\ \theta_{13} &\equiv \theta_{13}^{(0)} - \theta_z \quad \text{with} \quad \sin\theta_z \equiv -B\lambda \ , \end{aligned} \tag{48}$$

where the magnitudes of A and B are expected to be of $\mathcal{O}(1)$. In view of $\theta_{12}^{(0)} = 45^\circ$, $\theta_{23}^{(0)} \simeq 54.7^\circ$ and $\theta_{13}^{(0)} = 0^\circ$ given by U_0 together with $\theta_{12} \simeq 34^\circ$, $\theta_{23} \simeq 45^\circ$ and $\theta_{13} \simeq 9^\circ$ extracted from current neutrino oscillation data, for example, we typically obtain

$$\lambda \simeq 0.19 \ , \quad A \simeq 0.88 \ , \quad B \simeq 0.82 \ . \tag{49}$$

Up to the accuracy of $\mathcal{O}(\lambda^2)$, the sine and cosine of each flavor mixing angle are found to be

$$\begin{aligned} s_{12} &\simeq \frac{1}{\sqrt{2}}\left(1 - \lambda - \frac{1}{2}\lambda^2\right) \ , \\ c_{12} &\simeq \frac{1}{\sqrt{2}}\left(1 + \lambda - \frac{1}{2}\lambda^2\right) \ , \\ s_{13} &= B\lambda \ , \\ c_{13} &\simeq 1 - \frac{1}{2}B^2\lambda^2 \ , \\ s_{23} &\simeq \frac{\sqrt{2}}{\sqrt{3}}\left(1 - \frac{1}{\sqrt{2}}A\lambda - \frac{1}{2}A^2\lambda^2\right) \ , \\ c_{23} &\simeq \frac{1}{\sqrt{3}}\left(1 + \sqrt{2}A\lambda - \frac{1}{2}A^2\lambda^2\right) \ . \end{aligned} \tag{50}$$

Then the nine matrix elements of U can be expanded in terms of the small parameter λ as

$$
\begin{aligned}
U_{e1} &\simeq \frac{1}{\sqrt{2}}\left[1+\lambda-\frac{1}{2}\left(1+B^2\right)\lambda^2\right]e^{i\rho} \; , \\
U_{e2} &\simeq \frac{1}{\sqrt{2}}\left[1-\lambda-\frac{1}{2}\left(1+B^2\right)\lambda^2\right]e^{i\sigma} \; , \\
U_{e3} &= \hat{B}^*\lambda \; , \\
U_{\mu 1} &\simeq -\frac{1}{\sqrt{6}}\left\{1+\left(\sqrt{2}A-1+\sqrt{2}\hat{B}\right)\lambda-\frac{1}{2}\left[1+2\sqrt{2}A+A^2-2\left(\sqrt{2}-A\right)\hat{B}\right]\lambda^2\right\}e^{i\rho} \; , \\
U_{\mu 2} &\simeq \frac{1}{\sqrt{6}}\left\{1+\left(\sqrt{2}A+1-\sqrt{2}\hat{B}\right)\lambda-\frac{1}{2}\left[1-2\sqrt{2}A+A^2-2\left(\sqrt{2}+A\right)\hat{B}\right]\lambda^2\right\}e^{i\sigma} \; , \\
U_{\mu 3} &\simeq \frac{\sqrt{2}}{\sqrt{3}}\left[1-\frac{1}{\sqrt{2}}A\lambda-\frac{1}{2}\left(A^2+B^2\right)\lambda^2\right] \; , \\
U_{\tau 1} &\simeq \frac{1}{\sqrt{3}}\left\{1-\frac{1}{\sqrt{2}}\left(\sqrt{2}+A+\hat{B}\right)\lambda-\frac{1}{2}\left[1-\sqrt{2}A+A^2+\sqrt{2}\left(1+\sqrt{2}A\right)\hat{B}\right]\lambda^2\right\}e^{i\rho} \; , \\
U_{\tau 2} &\simeq -\frac{1}{\sqrt{3}}\left\{1+\frac{1}{\sqrt{2}}\left(\sqrt{2}-A+\hat{B}\right)\lambda-\frac{1}{2}\left[1-\sqrt{2}A+A^2+\sqrt{2}\left(1-\sqrt{2}A\right)\hat{B}\right]\lambda^2\right\}e^{i\sigma} \; , \\
U_{\tau 3} &\simeq \frac{1}{\sqrt{3}}\left[1+\sqrt{2}A\lambda-\frac{1}{2}\left(A^2+B^2\right)\lambda^2\right] \; , \qquad (51)
\end{aligned}
$$

where $\hat{B} \equiv Be^{i\delta}$ is defined, and the Majorana CP-violating phases ρ and σ are included. In this parametrization of U, the Jarlskog invariant of CP violation and two off-diagonal asymmetries defined in Eq. (47) turn out to be

$$
\mathcal{J}_l \simeq \frac{1}{3\sqrt{2}}B\lambda\sin\delta\left[1+\frac{1}{\sqrt{2}}A\lambda-\left(2A^2+B^2-2\right)\lambda^2\right] \; , \qquad (52)
$$

and

$$
\begin{aligned}
\mathcal{A}_{\rm L} &\simeq \frac{1}{3}\left[1-\sqrt{2}\left(\sqrt{2}+A+B\cos\delta\right)\lambda-\frac{1}{2}\left(A^2+5B^2-4\sqrt{2}A+2AB\cos\delta\right)\lambda^2\right] \; , \\
\mathcal{A}_{\rm R} &\simeq -\frac{1}{6}\left[1-2\left(2\sqrt{2}A-3\right)\lambda-\left(2A^2+B^2\right)\lambda^2\right] \; . \qquad (53)
\end{aligned}
$$

Taking $\lambda \simeq 0.19$ and $B \simeq 0.82$ for example, we obtain $\mathcal{J}_l/\mathcal{J}_{\rm max} \simeq \sqrt{6}B\lambda\sin\delta \sim 0.38\sin\delta$ in the leading-order approximation, consistent with our estimate made below Eq. (40). So the leptonic Jarlskog invariant can be as large as a few percent for an unsuppressed value of δ.

VIII. SUMMARY AND CONCLUDING REMARKS

Motivated by the robust Daya Bay result for a relatively large value of the smallest neutrino mixing angle θ_{13}, we have explored the leptonic flavor mixing structure and CP violation in a quite systematic way. Our main points and results are summarized as follows.

(1) We have outlined two phenomenological strategies for understanding the textures of lepton mass matrices and thus the structure of lepton flavor mixing:

- The MNSP matrix U is expressed as the sum of a constant leading term U_0 and a small perturbation term ΔU. U_0 is responsible for two larger mixing angles and may result from a certain flavor symmetry, while ΔU is responsible for the smallest mixing angle and CP-violating phase(s) and can be generated from the symmetry breaking or quantum corrections. As a consequence of the flavor symmetry at the level of lepton mass matrices, their entries have certain kinds of linear correlations or equalities.

- The mixing angles of U are associated with the ratios of charged-lepton and neutrino masses. In this case the lepton mass matrices may have some texture zeros which can also be derived from a certain flavor symmetry.

At present the first strategy is more popular for model building, but one has to come up with some new ideas in order to account for the observed value of θ_{13}. We stress that both approaches deserve further studies, in particular when the neutrino oscillation data on three flavor mixing angles become more and more precise.

(2) We have reexamined the democratic, bimaximal, tri-bimaximal, golden-ratio and hexagonal mixing patterns as possible candidates for U_0, and constrained their respective perturbations by using current experimental data. To generate $\theta_{13} \simeq 9^\circ$ together with the allowed values of θ_{12} and θ_{23}, we find that the structure of ΔU with respect to the democratic mixing pattern seems to be most natural because its nine elements are all of $\mathcal{O}(0.1)$. So we have proposed a Wolfenstein-like expansion of the MNSP matrix U with the help of the democratic mixing pattern and a small parameter $\lambda \simeq 0.19$, as compared with the well-known Wolfenstein parametrization of the CKM matrix V.

(3) Concentrating on the general conjecture $U = (U_0 + \Delta U)\, P_\nu$, we have discussed the possibly minimal form of ΔU for a given pattern of U_0 as mentioned above. The possibility of $(U_0)_{e3} \neq 0$ has also been taken into account. Let us emphasize two points in the following:

- Given $(U_0)_{e3} = 0$ (e.g., the tri-bimaximal mixing pattern), the ΔU part has to be taken more seriously than before in building a realistic model of lepton mass matrices. The reason is simply that it is a highly nontrivial job to generate $\theta_{13} \simeq 9^\circ$ from $\theta_{13}^{(0)} = 0^\circ$.

- It is worth paying more attention to the patterns of U_0 with nonzero $\theta_{13}^{(0)}$, such as the correlative or tetra-maximal mixing patterns. In this case one might be able to adjust the structure of ΔU to a simple form, but whether the origin of U_0 itself has a good reason (e.g., a simple or convincing flavor symmetry) remains an open question.

For a detailed analysis of the renormalization-group running effects on U with the value of θ_{13} as observed in the Daya Bay experiment, we refer the reader to Ref. [31].

(4) We have pointed out a salient feature of the MNSP matrix U: it may exhibit an approximate μ-τ permutation symmetry in modulus thanks to $\theta_{23} \simeq 45^\circ$. It is therefore crucial for the future neutrino oscillation experiments to determine the departure of θ_{23} from 45°. From the point of view of model building, the sign of $\theta_{23} - 45^\circ$ is a useful and sensitive model discriminator as the size of θ_{13} is.

(5) We have stressed that $\delta \simeq \pm 90^\circ$ is not only important for enhancing the strength of leptonic CP violation but also helpful for making the structure of U closer to its μ-τ symmetry limit. A geometrical description of CP violation has also been highlighted by considering the language of the leptonic unitarity triangles.

(6) We have summarized the main merits of nine topologically distinct parametrizations of U. Some of them turn out to be useful in revealing the features of lepton flavor mixing and CP violation. We have also introduced an alternative way to describe the MNSP matrix U — it is a Wolfenstein-like expansion of U based on the democratic mixing pattern.

Let us reiterate that the relative sizes of the nine elements of the MNSP matrix U cannot be completely fixed unless we have known $\theta_{23} > 45^\circ$ or $\theta_{23} < 45^\circ$ as well as the range of δ. With the help of the available experimental data and the unitarity of U, we find

$$|U_{e1}| > |U_{\mu 3}| \sim |U_{\tau 3}| > |U_{\mu 2}| \sim |U_{\tau 2}| > |U_{e2}| > |U_{\mu 1}| \sim |U_{\tau 1}| > |U_{e3}| \; , \tag{54}$$

where "$\sim$" implies that the relative magnitudes of $|U_{\mu i}|$ and $|U_{\tau i}|$ (for $i = 1, 2, 3$) remain undetermined at present. In comparison, the nine elements of the CKM matrix V are known to have the following hierarchy [56]:

$$|V_{tb}| > |V_{ud}| > |V_{cs}| \gg |V_{us}| > |V_{cd}| \gg |V_{cb}| > |V_{ts}| \gg |V_{td}| > |V_{ub}| \; . \tag{55}$$

We see that there is a striking similarity between the quark and lepton flavor mixing matrices: the smallest elements of both V and U appear in their respective top-right corners.

It is certainly impossible to make an exhaustive overview of all the problems associated with the leptonic flavor mixing structure and CP violation at this stage and in this paper [6]. But we hope that some of our points or questions may trigger some new ideas and further efforts towards deeper understanding of the underlying dynamics responsible for lepton mass generation, flavor mixing and CP violation. We emphasize that the lessons learnt from the quark sector are especially beneficial to our attempts in the lepton sector. Let us illustrate why this emphasis makes sense from a historical point of view as below.

In the history of flavor physics it took quite a long time to measure the four independent parameters of the CKM matrix V, but the experimental development had a clear roadmap:

$$\vartheta_{12} \text{ (or } |V_{us}|) \quad \Longrightarrow \quad \vartheta_{23} \text{ (or } |V_{cb}|) \quad \Longrightarrow \quad \vartheta_{13} \text{ (or } |V_{ub}|) \quad \Longrightarrow \quad \delta \text{ (quark)} \; . \tag{56}$$

Namely, the observation of the largest mixing angle ϑ_{12} was the first step, the determination of the smallest mixing angle ϑ_{13} (or equivalently, the smallest matrix element $|V_{ub}|$) was an important turning point, and then the quark flavor physics entered an era of precision measurements in which CP violation could be explored and new physics could be searched for. Interestingly and hopefully, the lepton flavor physics is repeating the same story:

$$\theta_{23} \text{ (or } |U_{\mu 3}|) \quad \Longrightarrow \quad \theta_{12} \text{ (or } |U_{e2}|) \quad \Longrightarrow \quad \theta_{13} \text{ (or } |U_{e3}|) \quad \Longrightarrow \quad \delta \text{ (lepton)} \; , \tag{57}$$

where θ_{23} is the largest and θ_{13} is the smallest. The observation of θ_{13} (or equivalently, the smallest matrix element $|U_{e3}|$) in the Daya Bay experiment is paving the way for future

[6] In particular, the impact of $\theta_{13} \simeq 9^\circ$ on the renormalization-group running behaviors of three flavor mixing angles and three CP-violating phases is not covered, nor is the issue for going beyond the 3×3 lepton flavor mixing matrix in the presence of three light or heavy sterile neutrinos. As for these two topics, we refer the reader to Ref. [31] and Ref. [40] respectively.

experiments to study leptonic CP violation and to look for possible new physics (e.g., whether the 3×3 MNSP matrix U is exactly unitary or not), in particular through the measurements of neutrino oscillations for different sources of neutrino beams. The Majorana nature of three massive neutrinos and their other two CP-violating phases (i.e., ρ and σ) can also be probed in the new era of neutrino physics.

ACKNOWLEDGMENTS

I would like to thank S. Luo for her technical helps, and to Y.F. Li and S. Luo for some useful discussions. This work was supported in part by the National Natural Science Foundation of China under grant No. 11135009.

REFERENCES

[1] Particle Data Group, K. Nakamura *et al.*, J. Phys. G **37**, 075021 (2010).

[2] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962); B. Pontecorvo, Sov. Phys. JETP **26**, 984 (1968).

[3] G.L. Fogli *et al.*, Phys. Rev. D **84**, 053007 (2011); T. Schwetz, M. Tortola, and J.W.F. Valle, New J. Phys. **13**, 109401 (2011).

[4] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Masakwa, Prog. Theor. Phys. **49**, 652 (1973).

[5] P.A.N. Machado *et al.*, arXiv:1111.3330.

[6] K. Abe *et al.* (T2K Collaboration), Phys. Rev. Lett. **107**, 041801 (2011).

[7] P. Adamson *et al.* (MINOS Collaboration), Phys. Rev. Lett. **107**, 181802 (2011).

[8] Y. Abe *et al.* (Double Chooz Collaboration), arXiv:1112.6353.

[9] Y.F. Wang, the news release and special seminar at IHEP, Beijing, 8 March 2012; F.P. An *et al.* (Baya Bay Collaboration), *Observation of electron antineutrino disappearance at Daya Bay*, submitted to Phys. Rev. Lett. (arXiv:1203.xxxx).

[10] H. Fritzsch and Z.Z. Xing, Phys. Lett. B **372**, 265 (1996); Phys. Lett. B **440**, 313 (1998); Phys. Rev. D **61**, 073016 (2000).

[11] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).

[12] F. Vissani, hep-ph/9708483; V. Barger, S. Pakvasa, T.J. Weiler, and K. Whisnant, Phys. Lett. B **437**, 107 (1998).

[13] P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. B **530**, 167 (2002); Z.Z. Xing, Phys. Lett. B **533**, 85 (2002); P.F. Harrison and W.G. Scott, Phys. Lett. B **535** (2002); X.G. He and A. Zee, Phys. Lett. B **560**, 87 (2003).

[14] Y. Kajiyama, M. Raidal, and A. Strumia, Phys. Rev. D **76**, 117301 (2007). A slight variation of this golden-ratio mixing pattern has been discussed by W. Rodejohann, Phys. Lett. B **671**, 267 (2009).

[15] Z.Z. Xing, J. Phys. G **29**, 2227 (2003); C. Giunti, Nucl. Phys. B (Proc. Suppl.) **117**, 24 (2003). The name of this flavor mixing pattern was coined by C.H. Albright, A. Dueck, and W. Rodejohann, Eur. Phys. J. C **70**, 1099 (2010).

[16] H. Fritzsch and Z.Z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000); and references therein.

[17] Z.Z. Xing, H. Zhang, and S. Zhou, Phys. Rev. D **77**, 113016 (2008); arXiv:1112.3112.

[18] H. Fritzsch and Z.Z. Xing, Phys. Lett. B **634**, 514 (2006); Phys. Lett. B **682**, 220 (2009).

[19] H. Fritzsch, Phys. Lett. B **73**, 317 (1978); Nucl. Phys. B **155**, 189 (1979).

[20] Z.Z. Xing, Phys. Lett. B **550**, 178 (2002); S. Zhou and Z.Z. Xing, Eur. Phys. J. C **38**, 495 (2005); H. Fritzsch, Z.Z. Xing, Y.L. Zhou, Phys. Lett. B **697**, 357 (2011).

[21] Z.Z. Xing, Phys. Lett. B **530**, 159 (2002); P.H. Frampton, S.L. Glashow, and D. Marfatia, Phys. Lett. B **536**, 79 (2002); Z.Z. Xing, Phys. Lett. B **539**, 85 (2002). For a systematic analysis, see: H. Fritzsch, Z.Z. Xing, and S. Zhou, JHEP **1109**, 083 (2011).

[22] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B **147**, 277 (1979).

[23] See, e.g., W. Grimus, A.S. Joshipura, L. Lavoura, and M. Tanimoto, Eur. Phys. J. C **36**, 227 (2004).

[24] See, e.g., R. Friedberg and T.D. Lee, High Energy Phys. Nucl. Phys. **30**, 591 (2006).

[25] For recent reviews with extensive references, see: G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010); L. Merlo, arXiv:1004.2211.

[26] Z.Z. Xing, Phys. Lett. B **696**, 232 (2011).
[27] W. Rodejohann, H. Zhang, and S. Zhou, Nucl. Phys. B **855**, 592 (2012); R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Nucl. Phys. B **858**, 437 (2012).
[28] Z.Z. Xing, Phys. Rev. D **78**, 011301 (2008).
[29] See, e.g., S. Antusch, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B **544**, 1 (2002); S. Antusch and M. Ratz, JHEP **0211**, 010 (2002); J.W. Mei and Z.Z. Xing, Phys. Rev. D **70**, 053002 (2004); Phys. Lett. B **623**, 227 (2005); J.W. Mei, Phys. Rev. D **71**, 073012 (2005); S. Luo and Z.Z. Xing, Phys. Lett. B **632**, 341 (2006); Phys. Lett. B **637**, 279 (2006); S. Goswami, S.T. Petcov, S. Ray, W. Rodejohann, Phys. Rev. D **80**, 053013 (2009); T. Araki, C.Q. Geng, and Z.Z. Xing, Phys. Lett. B **699**, 276 (2011); T. Araki and C.Q. Geng, JHEP **1109**, 139 (2011).
[30] H. Zhang and S. Zhou, Phys. Lett. B **704**, 296 (2011).
[31] S. Luo and Z.Z. Xing, in preparation.
[32] Z.Z. Xing and S. Zhou, Phys. Lett. B **653**, 278 (2007).
[33] Z.Z. Xing, H. Zhang, and S. Zhou, Phys. Lett. B **641**, 189 (2006).
[34] Z.Z. Xing, Chin. Phys. C **36**, 101 (2012); arXiv:1106.3244.
[35] M. Fukugita and T. Yanagida, Phys. Lett. B **174**, 45 (1986).
[36] Z.Z. Xing and S. Zhou, Phys. Lett. B **666**, 166 (2008).
[37] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).
[38] N. Cabibbo, Phys. Lett. B **72**, 333 (1978).
[39] W.L. Guo and Z.Z. Xing, Phys. Rev. D **66**, 097302 (2002).
[40] Z.Z. Xing, Phys. Rev. D **85**, 013008 (2012).
[41] Z.Z. Xing, Phys. Rev. D **63**, 073012 (2001).
[42] J.W. Mei and Z.Z. Xing, J. Phys. G **30**, 1243 (2004).
[43] H. Fritzsch and Z.Z. Xing, Nucl. Phys. B **556**, 49 (1999).
[44] S. Luo, Phys. Rev. D **85**, 013006 (2012).
[45] Z.Z. Xing, Int. J. Mod. Phys. A , 1 (2004); H. Zhang and Z.Z. Xing, Eur. Phys. J. C **41**, 143 (2005); Z.Z. Xing and H. Zhang, Phys. Lett. B **618**, 131 (2005).
[46] H. Fritzsch and Z.Z. Xing, Phys. Rev. D **57**, 594 (1998); Phys. Lett. B **517**, 363 (2001).
[47] R.P. Feynman, *The Nobel Lecture*, reprinted in Science **153**, 699 (1966).
[48] H. Fritzsch and Z.Z. Xing, Phys. Lett. B **413**, 396 (1997).
[49] Z.Z. Xing, Phys. Lett. B **633**, 550 (2006).
[50] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978); S.P. Mikheyev and A.Yu. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985).
[51] Y.L. Zhou, Phys. Rev. D **84**, 113012 (2011).
[52] S. Toshev, Mod. Phys. Lett. A **6**, 455 (1991).
[53] Z.Z. Xing, Phys. Rev. D **65**, 113010 (2002).
[54] J.M. Gerard and Z.Z. Xing, arXiv:1203.0496. See, also, J.M. Gerard, lectures given at the 2008 European School of High Energy Physics, Herbeumont, Belgium, June 2008 (arXiv:0811.0540).
[55] M. Huang *et al.*, arXiv:1108.3906.
[56] Z.Z. Xing, Nuovo Cim. A **109**, 115 (1996); J. Phys. G **23**, 717 (1997).

FIGURES

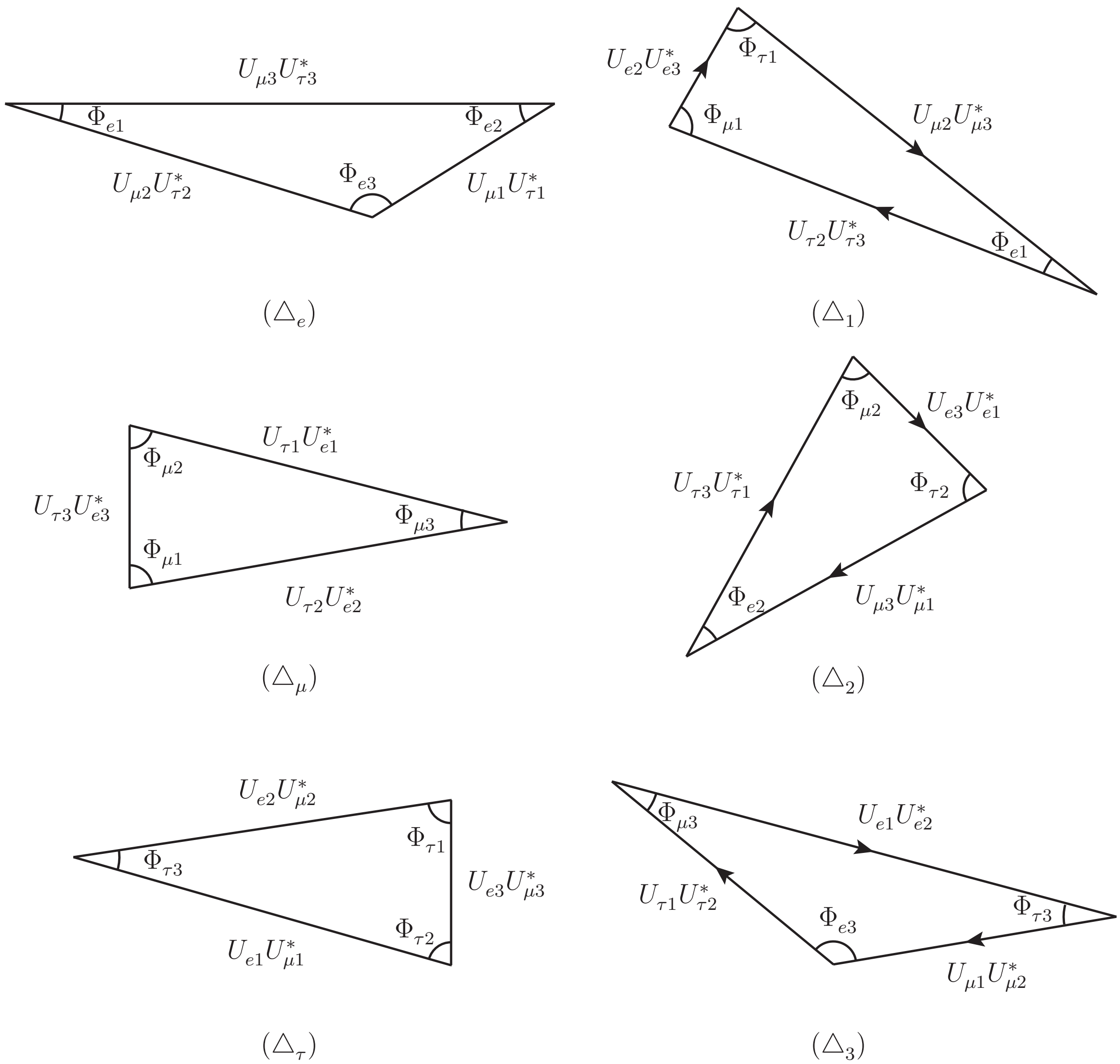

FIG. 1. Six unitarity triangles of the MNSP matrix U in the complex plane. Each triangle is named by the index that does not appear in its three sides [16], and the relative scale of the six triangles is roughly consistent with the assumption of $\delta \simeq 90^\circ$ and current experimental data on the three flavor mixing angles of U [41].

TABLES

TABLE I. A classification of nine topologically distinct parametrizations of the MNSP matrix U in terms of three rotation angles and three phase angles [43], where $P_\nu = \mathrm{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$ denotes the Majorana phase matrix. The phase (or sign) convention of each parametrization is adjustable.

Different parametrizations	Useful relations
Pattern (1): $U = R_{12}(\theta_{12}) \otimes R_{23}(\theta_{23}, \delta) \otimes R^T_{12}(\theta'_{12}) \otimes P_\nu$ $\begin{pmatrix} s_{12}s'_{12}c_{23} + c_{12}c'_{12}e^{-i\delta} & s_{12}c'_{12}c_{23} - c_{12}s'_{12}e^{-i\delta} & s_{12}s_{23} \\ c_{12}s'_{12}c_{23} - s_{12}c'_{12}e^{-i\delta} & c_{12}c'_{12}c_{23} + s_{12}s'_{12}e^{-i\delta} & c_{12}s_{23} \\ -s'_{12}s_{23} & -c'_{12}s_{23} & c_{23} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c_{12}s'_{12}c'_{12}s^2_{23}c_{23}\sin\delta$ $\tan\theta_{12} = \lvert U_{e3}/U_{\mu 3}\rvert$ $\tan\theta'_{12} = \lvert U_{\tau 1}/U_{\tau 2}\rvert$ $\cos\theta_{23} = \lvert U_{\tau 3}\rvert$
Pattern (2): $U = R_{23}(\theta_{23}) \otimes R_{12}(\theta_{12}, \delta) \otimes R^T_{23}(\theta'_{23}) \otimes P_\nu$ $\begin{pmatrix} c_{12} & s_{12}c'_{23} & -s_{12}s'_{23} \\ -s_{12}c_{23} & c_{12}c_{23}c'_{23} + s_{23}s'_{23}e^{-i\delta} & -c_{12}c_{23}s'_{23} + s_{23}c'_{23}e^{-i\delta} \\ s_{12}s_{23} & -c_{12}s_{23}c'_{23} + c_{23}s'_{23}e^{-i\delta} & c_{12}s_{23}s'_{23} + c_{23}c'_{23}e^{-i\delta} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s^2_{12}c_{12}s_{23}c_{23}s'_{23}c'_{23}\sin\delta$ $\cos\theta_{12} = \lvert U_{e1}\rvert$ $\tan\theta_{23} = \lvert U_{\tau 1}/U_{\mu 1}\rvert$ $\tan\theta'_{23} = \lvert U_{e3}/U_{e2}\rvert$
Pattern (3): $U = R_{23}(\theta_{23}) \otimes R_{13}(\theta_{13}, \delta) \otimes R_{12}(\theta_{12}) \otimes P_\nu$ $\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -c_{12}s_{23}s_{13} - s_{12}c_{23}e^{-i\delta} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\delta} & s_{23}c_{13} \\ -c_{12}c_{23}s_{13} + s_{12}s_{23}e^{-i\delta} & -s_{12}c_{23}s_{13} - c_{12}s_{23}e^{-i\delta} & c_{23}c_{13} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c_{12}s_{23}c_{23}s_{13}c^2_{13}\sin\delta$ $\tan\theta_{12} = \lvert U_{e2}/U_{e1}\rvert$ $\tan\theta_{23} = \lvert U_{\mu 3}/U_{\tau 3}\rvert$ $\sin\theta_{13} = \lvert U_{e3}\rvert$
Pattern (4): $U = R_{12}(\theta_{12}) \otimes R_{13}(\theta_{13}, \delta) \otimes R^T_{23}(\theta_{23}) \otimes P_\nu$ $\begin{pmatrix} c_{12}c_{13} & c_{12}s_{23}s_{13} + s_{12}c_{23}e^{-i\delta} & c_{12}c_{23}s_{13} - s_{12}s_{23}e^{-i\delta} \\ -s_{12}c_{13} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\delta} & -s_{12}c_{23}s_{13} - c_{12}s_{23}e^{-i\delta} \\ -s_{13} & s_{23}c_{13} & c_{23}c_{13} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c_{12}s_{23}c_{23}s_{13}c^2_{13}\sin\delta$ $\tan\theta_{12} = \lvert U_{\mu 1}/U_{e1}\rvert$ $\tan\theta_{23} = \lvert U_{\tau 2}/U_{\tau 3}\rvert$ $\sin\theta_{13} = \lvert U_{\tau 1}\rvert$
Pattern (5): $U = R_{31}(\theta_{13}) \otimes R_{12}(\theta_{12}, \delta) \otimes R^T_{13}(\theta'_{13}) \otimes P_\nu$ $\begin{pmatrix} c_{12}c_{13}c'_{13} + s_{13}s'_{13}e^{-i\delta} & s_{12}c_{13} & -c_{12}c_{13}s'_{13} + s_{13}c'_{13}e^{-i\delta} \\ -s_{12}c'_{13} & c_{12} & s_{12}s'_{13} \\ -c_{12}s_{13}c'_{13} + c_{13}s'_{13}e^{-i\delta} & -s_{12}s_{13} & c_{12}s_{13}s'_{13} + c_{13}c'_{13}e^{-i\delta} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s^2_{12}c_{12}s_{13}c_{13}s'_{13}c'_{13}\sin\delta$ $\cos\theta_{12} = \lvert U_{\mu 2}\rvert$ $\tan\theta_{13} = \lvert U_{\tau 2}/U_{e2}\rvert$ $\tan\theta'_{13} = \lvert U_{\mu 3}/U_{\mu 1}\rvert$
Pattern (6): $U = R_{12}(\theta_{12}) \otimes R_{23}(\theta_{23}, \delta) \otimes R_{13}(\theta_{13}) \otimes P_\nu$ $\begin{pmatrix} -s_{12}s_{23}s_{13} + c_{12}c_{13}e^{-i\delta} & s_{12}c_{23} & s_{12}s_{23}c_{13} + c_{12}s_{13}e^{-i\delta} \\ -c_{12}s_{23}s_{13} - s_{12}c_{13}e^{-i\delta} & c_{12}c_{23} & c_{12}s_{23}c_{13} - s_{12}s_{13}e^{-i\delta} \\ -c_{23}s_{13} & -s_{23} & c_{23}c_{13} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c_{12}s_{23}c^2_{23}s_{13}c_{13}\sin\delta$ $\tan\theta_{12} = \lvert U_{e2}/U_{\mu 2}\rvert$ $\sin\theta_{23} = \lvert U_{\tau 2}\rvert$ $\tan\theta_{13} = \lvert U_{\tau 1}/U_{\tau 3}\rvert$
Pattern (7): $U = R_{23}(\theta_{23}) \otimes R_{12}(\theta_{12}, \delta) \otimes R^T_{13}(\theta_{13}) \otimes P_\nu$ $\begin{pmatrix} c_{12}c_{13} & s_{12} & -c_{12}s_{13} \\ -s_{12}c_{12}c_{13} + s_{12}s_{13}e^{-i\delta} & c_{12}c_{23} & s_{12}c_{23}s_{13} + s_{23}c_{13}e^{-i\delta} \\ s_{12}s_{23}c_{13} + c_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} & -s_{12}s_{23}s_{13} + c_{23}c_{13}e^{-i\delta} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c^2_{12}s_{23}c_{23}s_{13}c_{13}\sin\delta$ $\sin\theta_{12} = \lvert U_{e2}\rvert$ $\tan\theta_{23} = \lvert U_{\tau 2}/U_{\mu 2}\rvert$ $\tan\theta_{13} = \lvert U_{e3}/U_{e1}\rvert$
Pattern (8): $U = R_{13}(\theta_{13}) \otimes R_{12}(\theta_{12}, \delta) \otimes R_{23}(\theta_{23}) \otimes P_\nu$ $\begin{pmatrix} c_{12}c_{13} & s_{12}c_{23}c_{13} - s_{23}s_{13}e^{-i\delta} & s_{12}s_{23}c_{13} + c_{23}s_{13}e^{-i\delta} \\ -s_{12} & c_{12}c_{23} & c_{12}s_{23} \\ -c_{12}s_{13} & -s_{12}c_{23}s_{13} - s_{23}c_{13}e^{-i\delta} & -s_{12}s_{23}s_{13} + c_{23}c_{13}e^{-i\delta} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c^2_{12}s_{23}c_{23}s_{13}c_{13}\sin\delta$ $\sin\theta_{12} = \lvert U_{\mu 1}\rvert$ $\tan\theta_{23} = \lvert U_{\mu 3}/U_{\mu 2}\rvert$ $\tan\theta_{13} = \lvert U_{\tau 1}/U_{e1}\rvert$
Pattern (9): $U = R_{13}(\theta_{13}) \otimes R_{23}(\theta_{23}, \delta) \otimes R^T_{12}(\theta_{12}) \otimes P_\nu$ $\begin{pmatrix} -s_{12}s_{23}s_{13} + c_{12}c_{13}e^{-i\delta} & -c_{12}s_{23}s_{13} - s_{12}c_{13}e^{-i\delta} & c_{23}s_{13} \\ s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ -s_{12}s_{23}c_{13} - c_{12}s_{13}e^{-i\delta} & -c_{12}s_{23}c_{13} + s_{12}s_{13}e^{-i\delta} & c_{23}c_{13} \end{pmatrix} P_\nu$	$\mathcal{J}_l = s_{12}c_{12}s_{23}c^2_{23}s_{13}c_{13}\sin\delta$ $\tan\theta_{12} = \lvert U_{\mu 1}/U_{\mu 2}\rvert$ $\sin\theta_{23} = \lvert U_{\mu 3}\rvert$ $\tan\theta_{13} = \lvert U_{e3}/U_{\tau 3}\rvert$